Stratified shear flow : from Rayleigh's theorem to Richardson's criterion (exam 2019)

We consider a stratified shear flow of velocity profile $U(z)\mathbf{e}_x$ and temperature profile $T(z)$ between two plates in $z = z_a$ and $z = z_b$, with $z_a < z_b$. We neglect the dependence along y, the transverse direction. We neglect viscous and thermal diffusion effects. The thermal expansion coefficient $\beta > 0$ connects the density variation with the temperature profile

$$\rho = \rho_0 - \beta\,(T - T_0)$$

We consider normal modes of the form $\exp(i(kx - \omega t))$. We denote the velocity components along x by $U + \epsilon u$, along z by $0 + \epsilon v$, the temperature $T + \epsilon\theta$ and pressure $P = 0 + \epsilon p$.

1. [1pt] In the case of constant temperature, what theorem justifies ignoring the transverse direction to determine the stability conditions of this flow?

2. [1pt] Show that the linearized equations write

$$\begin{aligned}
(-i\omega + ikU)u + \frac{dU}{dz}v &= -ik\frac{p}{\rho_0},\\
(-i\omega + ikU)v &= -\frac{1}{\rho_0}\frac{dp}{dz} + \frac{\beta g}{\rho_0}\theta,\\
(-i\omega + ikU)\theta + \frac{dT}{dz}v &= 0,\\
\frac{dv}{dz} + iku &= 0.
\end{aligned}$$

3. [1pt] What is the name of this last equation? What does it physically express?

4. [1pt] What is the name of the hypothesis which justifies its use while the density of the flow is not constant?

5. [1pt] We next assume that the base temperature profile a growing function of z i.e. $dT/dz = T' > 0$. Is the flow stably stratified? Do you expect any Rayleigh-Bénard instability to happen?

6. [3pts] We note $U'' = d^2U/dz^2$ and denote the derivation operator with respect to z by D. Define and interpret a classical auxiliary variable $\psi(z)\exp(i(kx - \omega t))$ to obtain the so-called Taylor-Goldstein equation

$$(-i\omega + ikU)\left(D^2 - k^2\right)\psi - ikU''\psi = \frac{k^2N^2}{-i\omega + ikU}\psi$$

   where N is the so-called Brunt-Väisälä frequency $N^2 = \frac{\beta g T'}{\rho_0}$.

7. [1pt] Check that, from a dimensional analysis point of view, N is indeed a frequency. To what limit equation discussed in class does the Taylor-Goldstein equation simply in absence of stratification when $N = 0$?

8. [1pt] Is this a classical or a polynomial eigenvalue problem?

9. [1pt] Separating the complex frequency in its real and imaginary part ($\omega = \omega_r + \ i\omega_i$), remind the condition for linear instability.

10. [3pts] Using a similar technique as that used in class, show that a necessary condition for the flow to be unstable is that there exist $z \in]z_a; z_b[$ such that

$$U'' \leq \frac{2\left(-\omega_r + kU\right)kN^2}{\omega_i^2 + \left(-\omega_r + kU\right)^2}.$$

N.B. As an intermediate step, you might first divide the Taylor-Goldstein equation by $(-i\omega + ikU)$, remember that $|\psi|^2 = \psi^*\psi$ (where * denotes complex conjugation) and that for any complex number $a = a_r + ia_i$, $1/a = \frac{a_r - ia_i}{a_r^2 + a_i^2}$. Why is that NOT a predictive criterion?

11. [2pts] BEWARE, this is a calculus-intensive question. Consider now the new variable

$$\chi = \frac{\psi}{\sqrt{-i\omega + ikU}}$$

Show that the Taylor-Goldstein equation becomes

$$D[(-i\omega + ikU)D\chi] + \left[k^2 \frac{U'^2/4 - N^2}{-i\omega + ikU} - \frac{ikU''}{2} - k^2(-i\omega + ikU)\right]\chi = 0$$

12. [3pts] By multiplying by $\chi*$ and integrating, show that a necessary conditions for the instability is given in terms of the Richardson number Ri is that $\exists y \in]z_a; z_b[$ such that

$$Ri = \frac{N(z)^2}{U'(z)^2} < \frac{1}{4}.$$

13. [1pt] What is the physical meaning behind this criterion? Is the temperature gradient stabilizing or destabilizing?

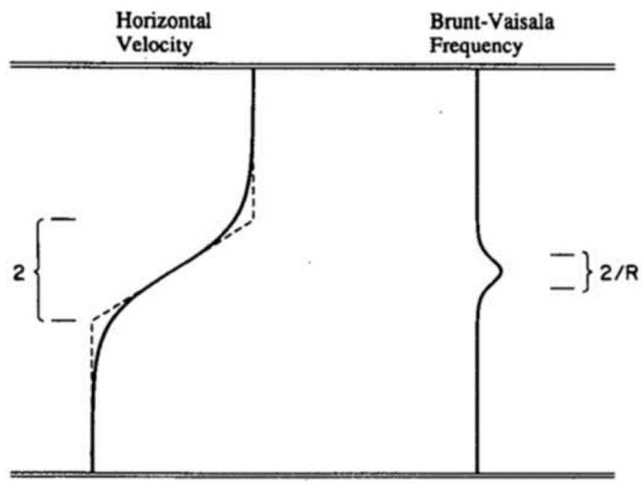

Figure 1: Solid curves represent the profile of velocity and background stratification for $R = 3$

14. [1pt] The dispersion relation of a velocity and temperature distribution represented in figure 1 (by its associated Brunt-Väisälä frequency N^2) have been computed by Smyth and Peltier in 1989. They have considered a localized temperature gradient of extension R, yielding

$$N^2(z) = J\left(1 - \tanh^2(Rz)\right),$$

where $J > 0$, while the velocity was chosen as

$$U(z) = \tanh(z).$$

Looking at the profile of N^2, comment on the stably or unstably stratified nature of the flow. Sketch a representative temperature distribution.

15. [1pt] The authors have used a numerical discretization method to determine the eigenvalues of the Taylor-Goldstein equation. With a central second order differential finite difference scheme and $N + 1$ points regularly spanning the interval $[z_a; z_b]$, how many interior points are there and how many eigenvalues are expected?

16. [1pt] We first consider $R = 1$, $Ri(z)$ is depicted in figure 2a for several values of J. Remembering the definition of the Richardson number $Ri(z)$, what is the value of its minimum $Ri_{\min}$ as a function of J ? Note that $\tanh'(z) = 1 - \tanh^2(z)$.

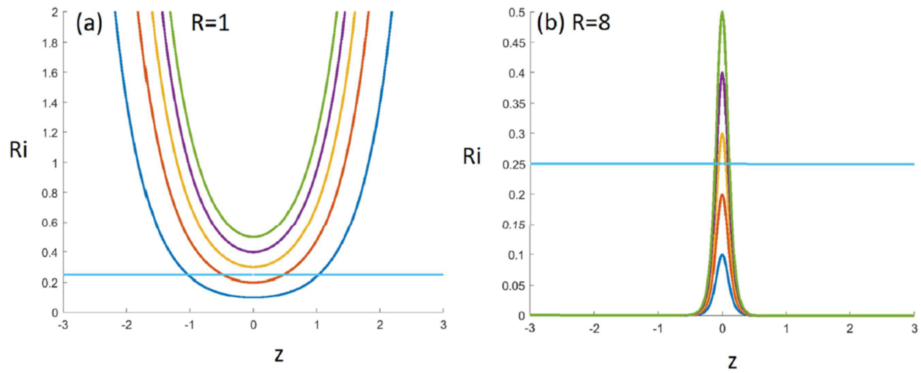

Figure 2: Richardson number $Ri(z)$ as a function of z for $J = 0.1$ (blue), $, \dots, 0.5$ (green) for $R = 1$ (a) and $R = 8$ (b). The 1/4-limit is also depicted.

17. [1pt] The isocontours of the growth-rate of the most unstable eigenvalues are depicted in figure 3a in the $k - J$ plane. Is the previously obtained necessary condition for instability violated?

18. [1pt] For $R = 8$, $Ri(z)$ is depicted in figure 2b for several values of J. For which values of J do you expect the flow to be unstable? The dominant eigenvalues have been computed by Smyth and Peltier (1989) in figure 4 for $R = 8$ and $J = 0.2, \dots, 0.8$. Do these results confirm your previous answer?

19. [1pt] Figure 3a refers to the Kelvin-Helmholtz instability. What is the consequence of increasing the strength of the stratification on its growing rates? Propose a physical interpretation.
In figure 4b, the two unstable branches associated to the Kelvin-Helmholtz instability switch to the so-called Holmboe instability for $k \geq\approx 0.25$. What is the consequence of increasing the strength of the stratification on Holmboe's growing rates? And on its most unstable wavenumber? Comparing figure 3a and 3b, what seems to be a necessary ingredient for an Holmboe instability to occur?

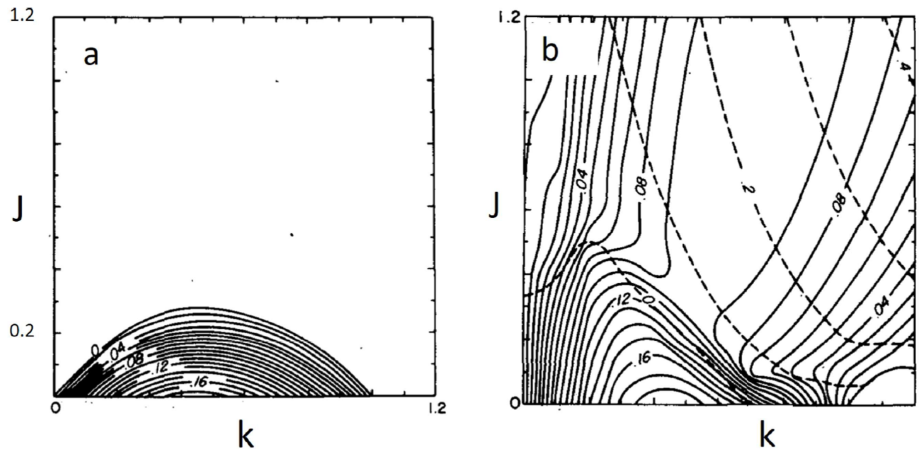

Figure 3: Isocontours of the growth-rate (full line) and frequency (dashed line) of the most unstable eigenvalues for $R = 1$ (a) and $R = 8$ (b).

20. [2pts] Consider the frequency curves on figure 4 (no dashed line means $\omega_r = 0$). From visual inspection only, can you predict if the Kelvin-Helmotz instability will be convective or absolute? Why? Same questions for Holmboe instabilities.

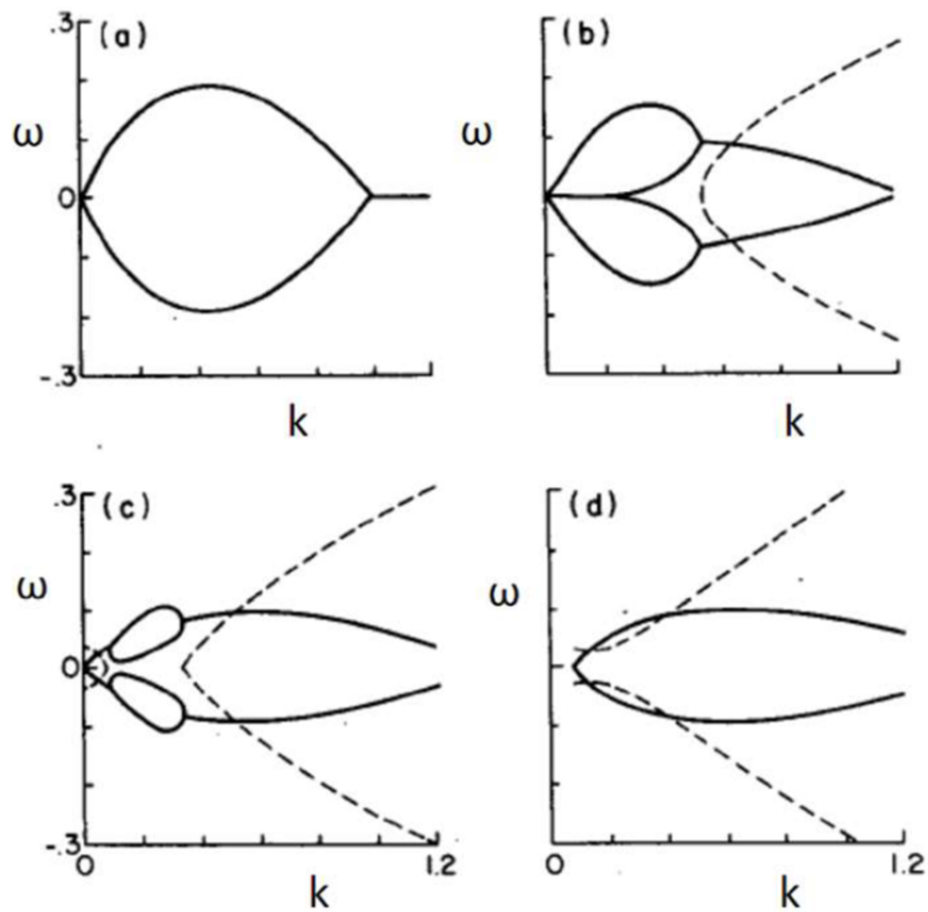

Figure 4: Growth-rate (full line) and frequency (dashed line) of the dominant unstable modes for $R = 8$ and different values of $J = 0.2, 0.4, 0.6$ and 0.8.